First order logic and twin-width in tournaments

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Twin-width is a graph invariant introduced by Bonnet et al. in 2020, with applications in algorithms, logic, and enumeration. A most notable one is a FPT algorithm for model checking of first order logic (FO) : given a graph G, a FO formula ϕ and a witness that G has twin-width t, one can test if G satisfies ϕ in time $f(\phi, t) \cdot |G|$, for some function f.

We prove that in tournaments, i.e. oriented complete graphs, the previous result is a characterisation of classes with efficient FO model checking. Precisely, a hereditary class of tournaments admits a FPT algorithm for FO model checking if and only if it has bounded twin-width. Furthermore, there is a FPT algorithm which approximates twin-width of tournaments up to some function. This builds upon and generalises a similar result of Bonnet et al. on ordered graphs [1].

The key tool used in this work is a variant of *binary search trees* (BST) for tournaments. The left-to-right order $<$ given by a BST on a tournament T has the crucial property that it always gives an approximation of twin-width, that is, the twin-width of the bi-relation $(T, <)$ is bounded by a function of the twin-width of T. Furthermore, if the twin-width of $(T, <)$ is very large, one can extract a subtournament $T' \subset T$ which still has large twin-width, and such that $<$ is definable inside T using a first-order transduction. Through these results on BST orders, we reduce the problem to the aforementioned result on ordered graphs.

These techniques further lead to a characterisation of bounded twin-width in tournaments through forbidden substructures, and show that a hereditary class of tournaments has bounded twin-width if and only if it is NIP, and if and only if its growth function is at most single-exponential. The results also generalise to oriented graphs with bounded independence number.

Références

[1] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width IV : ordered graphs and matrices. *CoRR*, abs/2102.03117, 2021, to appear at STOC 2022.