Erdős-Pósa property of holes in planar graphs

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For a graph G, let $\nu_{\text{cycles}}(G)$ denote the maximum number of vertex-disjoint cycles that can be found in G, and let $\tau_{\text{cycles}}(G)$ denote the minimum size of a vertex subset X meeting all cycles of G. A classic theorem of Erdős and Pósa states that there is a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\tau_{\text{cycles}}(G) \leq f(\nu_{\text{cycles}}(G))$$

for every graph G. It is also known that the function $f(k)$ can be taken to be in $O(k \log k)$ and this is best possible. However, it is known that a much better bound holds for planar graphs :

$$\tau_{\text{cycles}}(G) \leq 3 \cdot \nu_{\text{cycles}}(G)$$

for every planar graph G. It is even conjectured that the factor 3 can be reduced to 2, this is known as Jones's Conjecture ; this would be best possible.

Recently, Kim and Kwon considered a variant where instead of considering all cycles of G in the above definitions, we only consider those cycles of G that are induced and have length at least 4 ; such cycles are called *holes* of G. Defining $\nu_{\text{holes}}(G)$ and $\tau_{\text{holes}}(G)$ accordingly, Kim and Kwon proved that there is a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\tau_{\text{holes}}(G) \leq f(\nu_{\text{holes}}(G))$$

for every graph G, with a function $f(k)$ in $O(k^2 \log k)$. Our main result is the following linear bound for planar graphs :

$$\tau_{\text{holes}}(G) \leq 7 \cdot \nu_{\text{holes}}(G)$$

for every planar graph G. We also show with a construction that the constant factor 7 cannot be reduced below 3 ; in other words, the obvious adaptation of Jones's Conjecture to the setting of holes is not true.

Références

[1] Eun Jung Kim and O-Joung Kwon, *Erdős-Pósa property of chordless cycles and its applications*, J. Combin. Theory Ser. B **145** (2020), 65–112.