Packing signatures in signed graphs

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We define the *signature packing number* of a signed graph (G, σ), denoted $\rho(G, \sigma)$, to be the maximum number of signatures $\sigma_1, \sigma_2, \cdots, \sigma_l$ such that each σ_i is switching equivalent to σ and no edge is assigned a negative sign by more than one σ_i. In this talk, first in connection to recent developments on the theory of homomorphisms of signed graphs we show that given a signed graph (G, σ), $\rho(G, \sigma) \geq d + 1$ if and only if (G, σ) admits a homomorphism to SPC_d^o. Here SPC_d^o is a signed graph whose vertices are the elements of Z_2^d, x is adjacent to y with a positive edge if $x + y \in \{0, e_1, \cdots, e_d\}$ and it is adjacent with a negative edge if $x + y = J$, where $\{e_i\}$ is the standard bases and $J = (1, 1, \cdots, 1)$.

Observing that $\rho(G, \sigma) \leq g_-(G, \sigma)$, where $g_-(G, \sigma)$ is the length of the shortest negative closed walk of (G, σ), we study sufficient conditions under which equality holds. A particular conjecture in this regard is the following.

Conjecture. If (G, σ) is a planar connected signed graph with no positive odd-walk, then $\rho(G, \sigma) = g_-(G, \sigma)$.

The case of $g_-(G, \sigma) = 3$ is equivalent to the 4-color theorem. The general case is equivalent and related to some other conjectures. In this work, after further development of this theory of packing in signed graphs, we first give a short proof that the case $g_-(G, \sigma) = 4$ implies the case $g_-(G, \sigma) = 3$. We then sketch a proof of case $g_-(G, \sigma) = 4$ based on the 4-color theorem. The proof works on the larger class of K_5-minor-free graphs. More precisely we prove that :

Theorem. If G is a K_5-minor-free bipartite simple graph, then for any signature σ we have $\rho(G, \sigma) \geq 4$.